2009

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PG Kevrekidis
University of Massachusetts - Amherst, kevrekid@math.umass.edu

The higher-dimensional Ablowitz-Ladik model: from (non-)integrability and solitary waves to surprising collapse properties and more exotic solutions

P. G. Kevrekidis[1], G.J. Herring[2], S. Lafortune[3] and Q.E. Hoq[4]
[1] *Department of Mathematics and Statistics, University of Massachusetts, Amherst MA 01003-4515, USA*
[2] *Department of Mathematics and Statistics, Cameron University, Lawton, OK 73505, USA*
[3] *Department of Mathematics, College of Charleston, Charleston, SC 29401, USA*
[4] *Department of Mathematics and Computer Science, Western New England College, Springfield, MA, 01119, USA*

We propose a consideration of the properties of the two-dimensional Ablowitz-Ladik discretization of the ubiquitous nonlinear Schrödinger (NLS) model. We use singularity confinement techniques to suggest that the relevant discretization should not be integrable. More importantly, we identify the prototypical solitary waves of the model and examine their stability, illustrating the remarkable feature that near the continuum limit, this discretization leads to the *absence of collapse* and *complete spectral wave stability*, in stark contrast to the standard discretization of the NLS. We also briefly touch upon the three-dimensional case and generalizations of our considerations therein, and also present some more exotic solutions of the model, such as exact line solitons and discrete vortices.

arXiv:0907.1386v1 [nlin.PS] 8 Jul 2009

Introduction. The nonlinear Schrödinger (NLS) equation [1, 2] is a prototypical dispersive nonlinear partial differential equation (PDE) that has been central for almost four decades now to a variety of areas in mathematical physics. The relevant fields of application vary from optics and propagation of the (envelope of) the electric field in optical fibers [3, 4], to the self-focusing and collapse of Langmuir waves in plasma physics [5] and from the behavior of deep water waves and freak waves in the ocean [6, 7] to the mean-field dynamics of Bose-Einstein condensates in atomic physics [8, 9].

Much of the NLS literature and physical investigations have been centered around the one-dimensional (1d) setting, due not only to its mathematical and computational simplicity, but also in major part due to its complete integrability via the inverse scattering transform [2]. Yet, the higher dimensional investigations of the NLS equation have important elements of mathematics and physics in their own right, presenting the possibility for self-focusing and wave collapse [1] that has remarkable manifestations in some of the physical areas represented above. It is, thus, not surprising that recent fundamental investigations have been focused on related issues, such as e.g. the observation of the self-similarly collapsing solitary wave of the two-dimensional (2d) NLS equation (the so-called Townes soliton) [10], and on how to avoid the relevant collapse phenomena by means of temporal [11, 12] or spatial [13] variations of the nonlinearity, or by posing the problem on a lattice [14].

In the present work, we revisit the higher dimensional NLS equation and consider its properties on a spatial lattice. This is an interesting approach in its own right, at least in part because many of the problems associated with the optics (namely, optical waveguides [15]) or atomic physics (namely, Bose-Einstein condensates – BECs– in optical lattices [16]) are inherently associated with such discrete models. Another relevant motivation is that of computation, since even computing with the continuum model proper requires posing the problem on a computational grid (with a finite, but small spacing). However, in our investigations herein, we will not use the "canonical" discrete form of the NLS equation, the so-called DNLS model [17, 18]. Instead, we will consider an unconventional discretization that has yielded considerable insight (although, for a quite different reason, as illustrated below), namely the Ablowitz-Ladik (AL-NLS) model [2], which in its two-dimensional form reads:

$$\begin{aligned} i\dot{u}_{n,m} = &- \varepsilon\left(u_{n+1,m}+u_{n-1,m}+u_{n,m+1}+u_{n,m-1}-4u_{n,m}\right) \\ &+ \frac{\sigma}{4}|u|^2\left(u_{n+1,m}+u_{n-1,m}+u_{n,m+1}+u_{n,m-1}\right) \quad (1) \end{aligned}$$

In particular, the fundamental difference from the DNLS here is that in addition to the centered-difference approximation of the Laplacian (with $\varepsilon = 1/\Delta x^2$ being the coupling strength), a nearest-neighbor average is used to discretize the cubic nonlinearity of the continuum limit $\sigma|u|^2u$ (instead of a local term $\sigma|u_{n,m}|^2u_{n,m}$ in the DNLS). The cubic nonlinearity physically represents the Kerr effect in optics (i.e., the dependence of the refractive index of the optical material on the light intensity) or the mean-field approximation of interatomic interactions as a nonlinear self-action in BECs.

Our main findings and presentation are as follows. In the next section, we use the technique of singularity confinement to illustrate that, by analogy to the continuum model, the AL-NLS model is unlikely to be integrable in two spatial dimensions. We then embark on a systematic analysis of the model's solitary waves and their stability and substantiate a surprising result, namely that the AL-NLS discretization possesses *spectrally stable* solitons for arbitrarily small lattice spacings Δx. We then consider the generalization of this result in the 3d setting. Lastly, we present some more exotic solutions of the AL-NLS model, such as the analytically exact (but unstable) line soliton and the x-shaped discrete vortex.

Singularity Confinement. To determine whether the

AL-NLS is completely integrable, we use an integrability detector designed for difference or differential-difference equations, namely the singularity confinement (SC) criterion [19, 20]. The SC deals with the spontaneous appearance of a singularity at some point in the lattice. The criterion is satisfied if the singular behavior is confined in a finite region of the lattice. In order to apply the SC criterion, we re-interpret the AL-NLS as an iteration for the real and imaginary parts of $u = v + iw$:

$$v_{n+1,m} = \frac{4\dot{w}_{n,m} + 16\epsilon v_{n,m}}{4\epsilon - \sigma|u|^2} - v_{n,m+1} - v_{n-1,m} - v_{n,m-1},$$

$$w_{n+1,m} = \frac{-4\dot{v}_{n,m} + 16\epsilon w_{n,m}}{4\epsilon - \sigma|u|^2} - w_{n,m+1} - w_{n-1,m} - w_{n,m-1}.$$

If, at some point (n_0, m_0), the denominator above is zero for some time $t = t_0$, then the iterates $v_{n+1,m}$, $w_{n+1,m}$ will be singular. We thus set $4\epsilon - \sigma|u_{n_0,m_0}|^2 = (t-t_0)\alpha(t)$ (for an arbitrary function α nonzero at $t = t_0$), and let $v_{n,m}$ and $w_{n,m}$ for $n = n_0$ (with $m \neq m_0$) and $n = n_0 - 1$ to be regular and arbitrary. Computation of the iterates indicates that the singular behavior at $t = t_0$ is not confined but rather propagates indefinitely for larger values of n. This result implies that the AL-NLS is not completely integrable.

Solitary Waves and Stability. We start by recalling that the continuum analog of the 2d model is unstable (due to an instability which is weaker than exponential [21]) towards self-similar collapse [1]. The solitonic standing wave continuum solution (in the form $u(x, y, t) = \exp(i\Lambda t)v(x, y)$, taking $\Delta x \to 0$ and $\sigma = -1$ in Eq. (1)) is the so-called Townes soliton [22] with a squared L^2-norm (i.e., optical power or number of atoms in BEC) equal to $P_c \approx 11.7$. This is the separatrix between the collapse regime, arising for powers $P > P_c$, and dispersion occurring for $P < P_c$.

In the discrete case of the AL-NLS, we also seek standing waves in the form $u_{n,m}(t) = \exp(i\Lambda t)v_{n,m}$. Fixing $\sigma = -1$ (i.e., focusing nonlinearity), we can seek such solutions either as a function of Λ or (equivalently, up to a simple rescaling) as a function of Δx. We present both cases in Fig. 1. The continuum limit is obtained as Λ or, respectively, $\Delta x \to 0$. The former case has a particularity with respect to stability (which is part of the reason as to why it is presented herein). In particular, the so-called Vakhitov-Kolokolov criterion suggests that for *fundamental* solutions [1, 17, 18], the soliton stability is *solely* determined by the sign of the quantity $dP/d\Lambda$. In the AL-NLS case, interestingly (a direct generalization of the corresponding 1d conservation law [2]), the power is defined as:

$$P = \sum -\frac{\sigma}{4} \ln\left(1 - \frac{\sigma}{4\varepsilon}|u|^2\right), \tag{2}$$

such that it gives the continuum power in the limit. When, in our units, this quantity is positive, the solitary wave should be stable, while its negativity should imply exponential instability.

It is perhaps worthwhile to commence our comparison of the discrete solitary wave results to the corresponding continuum model through the intensely studied DNLS model [17, 18]. There it is known (see also the right panel of Fig. 1 for the relevant properties as a function of Δx) that as the continuum limit is approached, at the critical threshold of $\varepsilon = \Lambda$ (for spacings or frequencies below that), the solitary waves become *exponentially unstable*. This dynamical instability, manifested the right panels of Fig. 2 for two different values of the spacing Δx, is connected to the collapse of the continuum model and consists of a "quasi-collapse" phenomenon, whereby all the power of the solution is "collected" on a single site; the discrete model with its discrete power conservation and its spatial grid scale disallows a true self-similar collapse leading, in principle, to a finite time singularity. It should be noted that once unstable, the fundamental soliton of the DNLS model remains unstable throughout the parametric continuation to the continuum limit. The spectral picture of the linearization around the solitary wave (to examine the fate of small perturbations) reveals that in addition to a pair of eigenvalues always at the origin [due to the U(1) phase invariance of the model], there are 6 other pairs near the origin, approaching it, as the 2d continuum limit draws near. Of these, two identical pairs are associated with the translations along the x- and y-directions (whose corresponding invariances are restored in the limit) and which always remain stable (i.e., with $\lambda^2 < 0$). On the contrary, the eigenvalue pair with $\lambda^2 > 0$ connected with the slope condition and with the quasi-collapse (focusing instability in the language of [23]) is also connected with the additional symmetry of the 2d limit, namely the so-called pseudo-conformal invariance [1], which allows the self-similar reshaping of the solution (and, hence, the access to the collapse dynamics of the continuum limit).

One of the remarkable findings of the present work is that AL-NLS model are the *fundamentally* different spectral properties of its solitary wave. In particular, as can be seen in the top panel of Fig. 1, the change of sign of $dP/d\Lambda$ (and, accordingly, the instability) arises for $0.34 < \Lambda < 0.87$ for $\Delta x = 1$, or, respectively, the instability emerges for $0.4825 < \Delta x < 0.758$ (for $\Lambda = 1.5$). *However*, as the continuum limit is approached either through $\Lambda \to 0$, or through $\Delta x \to 0$, the relevant eigenvalue associated (in the limit) with the pseudo-conformal invariance has $\lambda^2 < 0$. Furthermore, the squared eigenvalues double pair associated with translations also approaches zero from the negative side. This implies that arbitrarily close to the limit, the AL-NLS discretization offers an alternative of NLS *free of collapse instabilities.* This feature is also evident in the case of the dynamical evolution of Fig. 2, whose top left panel illustrates that for small values of Δx, the solitary waveform is not subject to collapse (as it may be e.g. within the instability interval arising for higher Δx, as illustrated in the figure

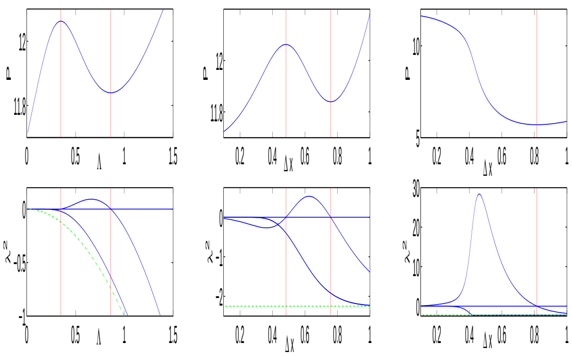

FIG. 1: (Left) Dependence of the AL-NLS fundamental soliton power P (Eq. (2)) and of the square eigenvalues of the linearization λ^2 on the wave frequency Λ ($\Delta x = 1$). Vertical red lines bound the instability region of $dP/d\Lambda < 0$. Green dashed line denotes the (lower) edge of the phonon band. (Middle) Same but for the variation as a function of the grid spacing Δx ($\Lambda = 1.5$). (Right) Same as the middle panel, but now for the regular DNLS discretization.

for $\Delta x = 0.6$).

Further consideration of this feature suggests that it is a *particular* trait of *critical* settings, which are at the very special (yet, physically realizable and important) separatrix between subcritical settings where the solitary waves are dynamically stable and supercritical ones, where the waves are exponentially unstable. In this critical case, the linear spectrum possesses an additional zero eigenvalue pair (associated with the pseudo-conformal invariance), which permits the reshaping of the solution under the action of the group of rescalings, and hence paves the way for the emergence of self-similar collapse. Discreteness can then shift this pair along the imaginary axis or along the real axis (as it is well documented to potentially do with respect to translational eigenvalues also [17, 18]). The AL-NLS discretization turns out to be a prototypical example whereby the eigenvalue pair formerly associated with the pseudo-conformal invariance is perturbed in a stable way (moves along the imaginary axis of the spectral plane), upon discretization and hence, this model allows infinitesimally small spacings to give rise to *collapse-free* dynamics.

Higher-dimensional Generalizations and More Complex Waveforms. The above discussion paves the way for understanding the approach to the continuum limit of the 3d solitary wave in the 3d AL-NLS generalization. The relevant power-frequency diagram is shown in Fig. 3, clearly illustrating through its negative slope the completely unstable approach to the limit (since $dP/d\Lambda < 0$). In this case, the continuum limit has $P \approx 18.82$ [24] and the soliton is *strongly* (exponentially) unstable due to the supercritical collapse. Hence, the discretization has no choice but to respect the relevant limit and thus infinitesimally small grid spacings are unable to provide

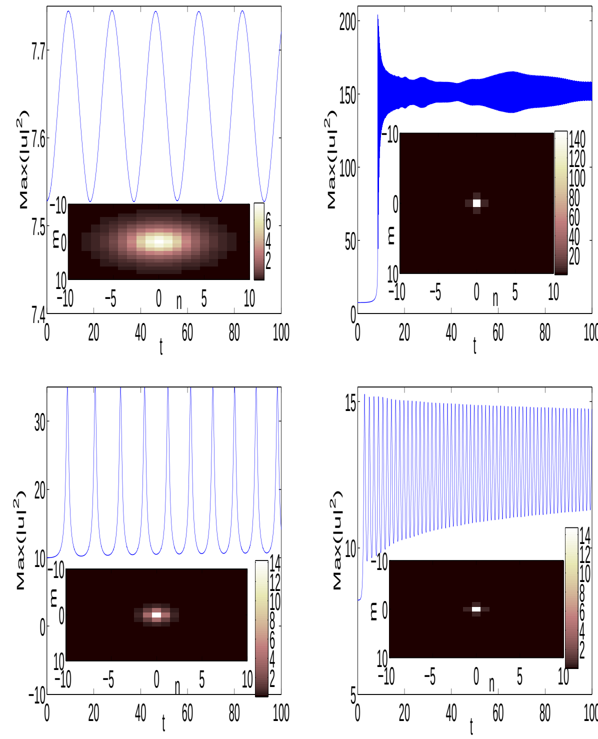

FIG. 2: Dynamical Evolution of the soliton solution maximum for AL-NLS (left) and DNLS (right) for $\Delta x = 0.2$ (top) and 0.6 (bottom). The insets show the contour plot of $|u_{n,m}(100)|^2$. Notice the stability of the AL-NLS for $\Delta x = 0.2$ (all other cases lead to collapse)

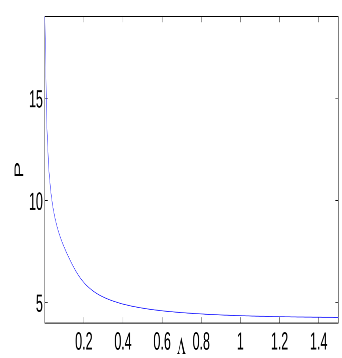

FIG. 3: Power vs. frequency plot for the unstable $dP/d\Lambda < 0$ 3d AL-NLS case ($\Delta x = 1$).

stabilization *irrespectively* of the specific form of the discrete model.

Lastly, we consider some more complex waveforms that can arise in the AL-NLS model. It is possible to obtain explicit solutions in the form of quasi-1d solitons e.g.,

$$u_{n,m} = A\mathrm{sech}(\alpha n + \beta m + x_0)\exp(i\Lambda t) \tag{3}$$

where the parameters satisfy $A = \pm\sqrt{(\Lambda^2 + 8\Lambda\varepsilon)/(4\varepsilon)}$, and $\alpha = \beta = \cosh^{-1}[(\Lambda + 4\varepsilon)/(4\varepsilon)]$. Such a solution

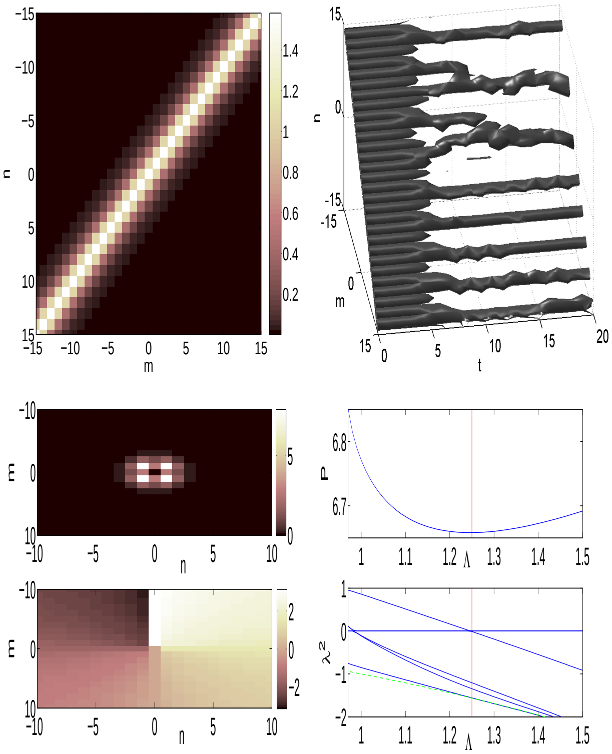

FIG. 4: The top panels show the exact line soliton contour (left) and its dynamical instability evolution (right); $\Lambda = 2\varepsilon = 1$. The bottom ones show the vortex square modulus and phase for $\Lambda = 1.1$ (left) and its power and linearization spectrum similar to Fig. 1 (right).

is depicted in the top left panel of Fig. 4, but the linearization around it illustrates that it is highly unstable and its dynamics spontaneously lead to filamentation and the formation of localized solitary waves of the type considered previously, as seen in the top right panel of Fig. 4. On the other hand, there also exist more complex solutions, such as the discrete x-shaped vortices of the bottom left panel of Fig. 4. Such solutions exist in the DNLS equation, because of its anti-continuum limit $\varepsilon = 0$ [25] but do not persist in the continuum limit, hence their existence is not guaranteed in the AL-NLS model. Nevertheless, we find here that they exist and are quite robust, becoming unstable due to a real eigenvalue pair for $\Delta x < 1.25$ (bottom right panel of Fig. 4). Additional eigenvalue pairs emerge for $\Delta x < 0.985$. It is interesting to note that the corresponding x-shaped vortex in the DNLS case becomes unstable due to complex eigenvalue quartets for spacing values $\Delta x < 1.178$.

Conclusions. In this study, we considered the Ablowitz-Ladik discretization of the NLS in higher dimensional settings. We illustrated, via singularity confinement, that the model is unlikely to be integrable, yet, due to the critical nature of the nonlinearity in 2d, it possesses some remarkable features, including robust and spectrally stable solitary waves even infinitesimally close to the continuum limit, explicit analytical solutions and more complex vortex waveforms. This study raises many fundamental issues. It would be relevant to obtain a systematic understanding of how different NLS discretizations in the critical case affect the pseudo-conformal invariance (and whether there might conceivably exist one that preserves the symmetry). It would also be useful to examine how different types of discretizations affect other classes of critical models, such as the critical generalized KdV [26]. Lastly, it would be relevant to obtain a systematic classification (analogous to the one existing in DNLS [25]) of the solutions of the AL-NLS and their stability properties, in two- and higher dimensions.

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